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Diffusion in inhomogeneous media

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We consider the transport of conserved charges in spatially inhomogeneous quantum systems with a discrete lattice symmetry. We analyze the retarded two-point functions involving the charges and the associated currents at long wavelengths, compared to the scale of the lattice, and, when the dc conductivities are finite, extract the hydrodynamic modes associated with diffusion of the charges. We show that the dispersion relations of these modes are related to the eigenvalues of a specific matrix constructed from the dc conductivities and certain thermodynamic susceptibilities, thus obtaining generalized Einstein relations. We illustrate these general results in the specific context of relativistic hydrodynamics where translation invariance is broken using spatially inhomogeneous and periodic deformations of the stress tensor and the conserved $U(1)$ currents. Equivalently, this corresponds to considering hydrodynamics on a curved manifold, with a spatially periodic metric and chemical potential, and we obtain the dispersion relations for the heat and charge diffusive modes.

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I. INTRODUCTION

Motivated by various strongly correlated states of matter seen in nature, there has been a significant effort devoted to obtaining a deeper theoretical understanding of thermoelectric transport. It has long been appreciated that it is necessary to work within a framework in which momentum is not conserved. Indeed, for a translationally invariant system in which momentum is exactly conserved, the ac thermal response necessarily contains a delta function at zero frequency leading to a nonphysical infinite dc thermal conductivity. Thus, one is interested in setups in which translation symmetry is explicitly broken.

In this paper we will present some general results for thermoelectric transport in inhomogeneous systems. More precisely, we will consider general quantum systems, with one or more conserved currents, with a discrete, spatial lattice symmetry. This could describe, for example, a quantum field theory in which translation invariance has been explicitly broken by deforming the theory with operators which have a periodic dependence on the spatial coordinates.

Of central interest are the retarded Green's functions for the current-current correlators $G_{JJ}(t,\mathbf{x};t'\mathbf{x}')$. At the level of linear response these determine how the currents respond after perturbing the system by a current source. Time translation invariance implies that these Green's functions only depend on $t-t'$ which allows us to Fourier transform and obtain $G_{JJ}(\omega,\mathbf{x},\mathbf{x}')$. In a translationally invariant setting the Green's functions would also only depend on $\mathbf{x}'-\mathbf{x}$ and a Fourier transform leads to a correlator depending on ω and a single wave vector $\mathbf{k}$. When translations are broken, this is no longer possible but a discrete lattice symmetry allows us to define an infinite discrete set of correlators $G_{JJ}^{(\{n_i\})}(\omega,\mathbf{k})$, where $\{n_i\}$ are a set of integers. We will be particularly interested in studying the correlator $G_{JJ}(\omega,\mathbf{k})\equiv G_{JJ}^{(\{0\})}(\omega,\mathbf{k})$. Indeed this correlator, which satisfies a simple positivity condition, captures the transport properties of the system at late times and for wavelengths much longer than the scale of the lattice, and thus we might call $G_{JJ}(\omega,\mathbf{k})$ a "hydrodynamic-mode correlator."

By generalizing similar computations presented in [1] in the translationally invariant setting, we will show that when the thermoelectric dc conductivity is finite, subject to some analyticity assumptions, there is necessarily a diffusion pole in the hydrodynamic-mode correlator for $\omega,\varepsilon\mathbf{k}\to 0$. If the system has just a single conserved current then we explain precisely when we get a dispersion relation for the diffusion pole of the form

$$\omega=-i\varepsilon^2 D(\mathbf{k})+\ldots,\qquad D(\mathbf{k})=[\sigma_{dc}^{ij}k_ik_j]\chi(\mathbf{0})^{-1},\tag{1.1}$$

where σ_{dc}^{ij} is the dc conductivity and $\chi(\mathbf{k})$ is the charge susceptibility. This is our first Einstein relation for inhomogeneous media.

When there are additional conserved currents, there will be additional diffusion modes when the associated dc conductivities are finite. We analyze the dispersion relations for the diffusion modes and show how they can be obtained from the eigenvalues of a specific "generalized diffusion matrix" that is constructed from the dc conductivities and various thermodynamic susceptibilities. We emphasize that, generically, the dispersion relations for the diffusion modes are not of the form (1.1) and hence we refer to our result concerning the dispersion relation as a "generalized Einstein relation." This feature of diffusion modes was also emphasized in [2] within a specific hydrodynamic setting, which we will return to later.

These results concerning hydrodynamic modes of the Green's functions are very general. However, motivated by recent experimental progress [3–5], there has been considerable theoretical work using hydrodynamics to study thermoelectric transport [2,6–18] and it is therefore of interest to see how our general results on diffusion manifest themselves in this particular context. More specifically, we will study this within the context of relativistic hydrodynamics, describing the hydrodynamic limit of a relativistic quantum field theory.

Within this hydrodynamic framework, we first need to consider how momentum dissipation is to be incorporated. A standard approach is to modify, by hand, the hydrodynamic equations of motion, i.e. the Ward identities of the underlying field theory, by a phenomenological term that incorporates momentum dissipation (e.g. [19,20]). An alternative and more controlled approach is to maintain the Ward identities, which are fundamental properties of the field theory, but to consider the field theory to be deformed by spatially dependent sources. In this spirit, the hydrodynamic limit of a class of field theories which have been deformed by certain scalar operators was analyzed in [7]. Subsequently, the universal class of deformations which involve adding spatially dependent sources for the stress tensor were studied in [14]. Since the stress tensor of the field theory couples to the spacetime metric, the deformations studied in [14] are equivalent to studying the hydrodynamic limit of the quantum field theory on a curved spacetime manifold. The spacetime metric is taken to have a timelike Killing vector in order to discuss thermal equilibrium. Then, while spatial momentum will, generically, no longer be conserved, energy still will be. It may be possible to experimentally realize the deformations studied in [14] in real materials, such as strained graphene [21–23].

In this paper we extend the analysis of [14] to cover relativistic quantum field theories which have a conserved $U(1)$ symmetry. As in [14] we can consider the field theory to live on a static, curved manifold. Although not necessary, it will be convenient to take the manifold to have planar topology and with a metric that is periodic in the spatial directions. Within the hydrodynamic framework we will also consider deformations that are associated with spatially dependent sources for the $U(1)$ symmetry. This is particularly interesting since it corresponds to allowing for spatially dependent chemical potential or, equivalently, spatially dependent charge density. One can anticipate that our results will be useful for understanding thermoelectric transport in real systems, such as charged puddles, with or without strain, in graphene [24–27] as also discussed in [18].

As an application of our formalism, we show how to construct long-wavelength, late-time hydrodynamic modes that are associated with diffusion of both energy and electric charge. We derive the dispersion relation for these modes and explicitly obtain the generalized Einstein relations. It is worth noting that this result is independent of the precise transport coefficients that enter the constitutive relations in the conserved currents. We also note that a derivation of an Einstein relation for the diffusion of electric charge in the context of hydrodynamics with vanishing local charge density in one spatial direction was carried out in Appendix A of [13] and this is consistent with our more general analysis here.

II. GREEN'S FUNCTION PERSPECTIVE

We begin our discussion with a general quantum system with a time-independent Hamiltonian H. We assume that there is a lattice symmetry group which acts on the d spatial coordinates via $\mathbf{x} \to \mathbf{x} + \mathbf{L}_j$ and $U^{-1}_{\mathbf{L}_j} A(t,\mathbf{x}) U_{\mathbf{L}_j} = A(t, \mathbf{x} + \mathbf{L}_j)$, where $A(t,\mathbf{x})$ is an arbitrary local operator. We assume that the Hamiltonian is invariant under this symmetry and hence $U^{-1}_{\mathbf{L}_j} H U_{\mathbf{L}_j} = H$. We will also consider the system to be at finite temperature T.

As usual, for two local operators $A(t,\mathbf{x})$, $B(t,\mathbf{x})$, the retarded two-point functions are defined through

$$G_{AB}(t,\mathbf{x};t',\mathbf{x}') = -i\theta(t-t')\langle[A(t,\mathbf{x}), B(t',\mathbf{x}')]\rangle, \qquad (2.1)$$

with $\langle A(t,\mathbf{x})\rangle = \mathrm{Tr}(\rho A(t,\mathbf{x}))$, where $\rho = e^{-\beta H}/\mathrm{Tr}(e^{-\beta H})$ and $\beta = 1/T$. Using the fact that $A(t,\mathbf{x}) = e^{itH} A(0,\mathbf{x}) e^{-itH}$ and the lattice symmetry of H, we see that the two-point functions will satisfy

$$G_{AB}(t,\mathbf{x};t',\mathbf{x}') = G_{AB}(t-t',\mathbf{x};0,\mathbf{x}'), \qquad (2.2)$$

$$G_{AB}(t,\mathbf{x}+\mathbf{L}_j;t',\mathbf{x}'+\mathbf{L}_j) = G_{AB}(t,\mathbf{x};t',\mathbf{x}'). \qquad (2.3)$$

The symmetry (2.2) allows us to define a function with three arguments through $G_{AB}(t-t',\mathbf{x},\mathbf{x}') \equiv G_{AB}(t,\mathbf{x};t',\mathbf{x}')$.

We next recall that if we introduce a perturbative source term in the Hamiltonian via

$$\delta H(t) = \int d\mathbf{x}\, \delta h_B(t,\mathbf{x}) B(t,\mathbf{x}), \qquad (2.4)$$

then at the level of linear response, the change in the expectation values of an arbitrary operator A is given by

$$\delta\langle A\rangle(t,\mathbf{x}) = \int dt' d\mathbf{x}' G_{AB}(t-t',\mathbf{x},\mathbf{x}')\delta h_B(t',\mathbf{x}'). \qquad (2.5)$$

We note that the source, and hence the response, need not be a periodic function of the spatial coordinates and indeed this will be case of most interest in the following.

To proceed we Fourier transform the Green's function on all arguments and define

$$G_{AB}(\omega,\mathbf{k},\mathbf{k}')\equiv\int dtd\mathbf{x}d\mathbf{x}'e^{i\omega t-i\mathbf{k}\mathbf{x}+i\mathbf{k}'\mathbf{x}'}G_{AB}(t,\mathbf{x},\mathbf{x}'). \tag{2.6}$$

The discrete symmetry (2.3) implies that we can perform a crystallographic type of decomposition to obtain

$$G_{AB}(\omega,\mathbf{k},\mathbf{k}')=\sum_{\{n_j\}}G_{AB}^{(\{n_j\})}(\omega,\mathbf{k}')\delta(\mathbf{k}-\mathbf{k}'-n_j\mathbf{k}_L^j), \tag{2.7}$$

where $\mathbf{k}_L^j$ are the reciprocal lattice vectors satisfying $\mathbf{k}_L^i\cdot\mathbf{L}^j=2\pi\delta^{ij}$ and $\{n_j\}$ are sets of integers. To see this, we simply notice that if we define the function

$$G_{AB}(\omega,\mathbf{x},\mathbf{k}')\equiv\int dtd\mathbf{x}'e^{i\omega t+i\mathbf{k}'\mathbf{x}'}G_{AB}(t,\mathbf{x},\mathbf{x}'), \tag{2.8}$$

then the real space lattice symmetry (2.3) implies the periodicity condition

$$G_{AB}(\omega,\mathbf{x}+\mathbf{L}_j,\mathbf{k}')=e^{i\mathbf{k}'\mathbf{L}_j}G_{AB}(\omega,\mathbf{x},\mathbf{k}'), \tag{2.9}$$

and hence we can deduce that $e^{-i\mathbf{k}'\mathbf{x}}G_{BB'}(\omega,\mathbf{x},\mathbf{k}')$ is periodic as a function of $\mathbf{x}$. This lets us write it as a discrete Fourier series, expressing

$$G_{AB}(\omega,\mathbf{x},\mathbf{k}')=\frac{1}{(2\pi)^d}e^{i\mathbf{k}'\mathbf{x}}\sum_{\{n_j\}}e^{in_j\mathbf{k}_L^j\mathbf{x}}G_{AB}^{(\{n_j\})}(\omega,\mathbf{k}'), \tag{2.10}$$

and (2.7) follows.

In the sequel, we will be particularly interested in the zero modes, $G_{AB}(\omega,\mathbf{k})\equiv G_{AB}^{(\{0\})}(\omega,\mathbf{k})$. These can easily be obtained by taking average spatial integrals over a period of periodic functions. If we define $\oint\equiv(\prod L_i)^{-1}\int_{\{0\}}^{\{\mathbf{L}_i\}}d\mathbf{x}$ then we have

$$\begin{aligned}G_{AB}(\omega,\mathbf{k})&\equiv G_{AB}^{(\{0\})}(\omega,\mathbf{k})\\&=\oint d\mathbf{x}\int d\mathbf{x}'G_{AB}(\omega,\mathbf{x},\mathbf{x}')e^{i\mathbf{k}(\mathbf{x}'-\mathbf{x})}.\end{aligned} \tag{2.11}$$

From (2.6) we can also write

$$G_{AB}(\omega,\mathbf{k})=(N\prod_i L_i)^{-1}G_{AB}(\omega,\mathbf{k},\mathbf{k}), \tag{2.12}$$

where N is the total number of spatial periods in the system.

We next examine the positivity of the spectral weight of our operators. Working in the interaction picture, the system absorbs energy at rate

$$\frac{d}{dt}W(t)=\int d\mathbf{x}\delta\langle B\rangle(t,\mathbf{x})\frac{d}{dt}\delta h_B(t,\mathbf{x}), \tag{2.13}$$

where a summation over B is understood. Introducing the notation

$$\delta h_B(t,\mathbf{x})=\frac{1}{(2\pi)^{d+1}}\int d\omega d\mathbf{k}\delta h_B(\omega,\mathbf{k})e^{-i\omega t+i\mathbf{k}\mathbf{x}}, \tag{2.14}$$

we can show that the total energy absorbed by the system is

$$\begin{aligned}\Delta W=&-\frac{1}{(2\pi)^{2d+1}}\int d\omega d\mathbf{k}d\mathbf{k}'\delta h_B^*(\omega,\mathbf{k})\omega[\mathrm{Im}G]_{BB'}\\&\times(\omega,\mathbf{k},\mathbf{k}')\delta h_{B'}(\omega,\mathbf{k}'),\end{aligned} \tag{2.15}$$

where $[\mathrm{Im}G]_{AB}(\omega,\mathbf{k},\mathbf{k}')\equiv\frac{1}{2i}[G_{AB}(\omega,\mathbf{k},\mathbf{k}')-G_{BA}^*(\omega,\mathbf{k}',\mathbf{k})]$. To get to the last line we used $G_{AB}(\omega,\mathbf{k},\mathbf{k}')=G_{AB}(-\omega,-\mathbf{k},-\mathbf{k}')^*$ (for real frequencies and wave vectors), which follows from the reality of $G_{AB}(t,\mathbf{x},\mathbf{x}')$. Since $\delta h_B(\omega,\mathbf{k})$ are arbitrary we deduce that $-\omega[\mathrm{Im}G]_{AB}(\omega,\mathbf{k},\mathbf{k}')$ is a positive semidefinite matrix, with matrix indices including both the operator labels as well as the wave vectors. Since the block diagonal elements of a positive semidefinite matrix are positive semidefinite, using (2.12) we can conclude that the zero modes $-\omega ImG_{AB}(\omega,\mathbf{k})$ are positive semidefinite. In particular we have

$$-\omega ImG_{AA}(\omega,\mathbf{k})\geq 0, \tag{2.16}$$

with no sum on A. The positive semidefinite aspect of $-\omega[\mathrm{Im}G]_{AB}(\omega,\mathbf{k},\mathbf{k}')$ also gives rise to additional conditions for the $G_{AB}^{(\{n_j\})}(\omega,\mathbf{k})$, with $\{n_j\}\neq\{0\}$.

To conclude this subsection we examine how the Green's functions behave under time reversal invariance. For simplicity we will assume that the periodic system is invariant under time reversal. Recall that this acts on local operators according to $TA(t,\mathbf{x})T^{-1}=\epsilon_A A(-t,\mathbf{x})$, where $\epsilon_A=\pm1$. Since T is an antiunitary operator we can deduce that $G_{AB}(t,\mathbf{x},\mathbf{x}')=\epsilon_A\epsilon_B G_{BA}(t,\mathbf{x}',\mathbf{x})$. Thus, we have $G_{AB}(\omega,\mathbf{k},\mathbf{k}')=\epsilon_A\epsilon_B G_{BA}(\omega,-\mathbf{k}',-\mathbf{k})$ and hence

$$G_{AB}^{(\{n_j\})}(\omega,\mathbf{k})=\epsilon_A\epsilon_B G_{BA}^{(\{n_j\})}(\omega,-\mathbf{k}-n_l\mathbf{k}_L^l). \tag{2.17}$$

Returning to the linear response given in (2.5), after taking suitable Fourier transforms we can write

$$\begin{aligned}&\delta\langle A\rangle(\omega,\mathbf{x})\\&=\frac{1}{(2\pi)^{2d}}\int d\mathbf{k}\sum_{\{n_j\}}e^{i(\mathbf{k}+n_j\mathbf{k}_L^j)\mathbf{x}}G_{AB}^{(\{n_j\})}(\omega,\mathbf{k})\delta h_B(\omega,\mathbf{k}).\end{aligned} \tag{2.18}$$

If we consider a source which contains a single spatial Fourier mode $\delta h_B(t,\mathbf{x})=e^{i\mathbf{k}_s\mathbf{x}}\delta h_B(t)$, then we have

$$\delta\langle A\rangle(\omega,\mathbf{x}) = e^{i\mathbf{k}_s\mathbf{x}}\sum_{\{n_j\}}\frac{1}{(2\pi)^d}e^{in_j\mathbf{k}_L^j\mathbf{x}}G_{AB}^{(\{n_j\})}(\omega,\mathbf{k}_s)\delta h_B(\omega),$$
$$\equiv e^{i\mathbf{k}_s\mathbf{x}}\sum_{\{n_j\}}e^{in_j\mathbf{k}_L^j\mathbf{x}}\delta\langle A\rangle^{(\{n_j\})}(\omega,\mathbf{k}_s). \tag{2.19}$$

Notice, in particular, that the zero mode in the summation is fixed by the zero mode of the Green's function: $\delta\langle A\rangle^{(\{0\})}(\omega,\mathbf{k}_s) = (2\pi)^{-d}G_{AB}(\omega,\mathbf{k}_s)\delta h_B(\omega)$.

In the next subsections we will take A and B to be components of conserved currents. In this context the zero-mode correlator $G_{AB}(\omega,\mathbf{k})$ captures transport of the associated hydrodynamic modes and hence one can call it a hydrodynamic-mode correlator.

A. Einstein relation for a single current

We now consider the operator A to be a current density[1] operator J^μ, which satisfies a continuity equation of the form $\partial_\mu J^\mu = 0$. From the definition (2.6) we have

$$-i\omega G_{J^tB}(\omega,\mathbf{k},\mathbf{k}') + i\mathbf{k}_iG_{J^iB}(\omega,\mathbf{k},\mathbf{k}') = 0, \tag{2.20}$$

for any operator B, whose equal time commutator with J^t vanishes. Using the crystallographic decomposition (2.7) in (2.20) we then have

$$-i\omega G_{J^tB}^{(\{n_j\})}(\omega,\mathbf{k}) + i(\mathbf{k}+n_j\mathbf{k}_L^j)_iG_{J^iB}^{(\{n_j\})}(\omega,\mathbf{k}) = 0. \tag{2.21}$$

We now[2] focus on the hydrodynamic-mode correlators with $\{n_j\} = 0$, which satisfy a positivity property discussed just above (2.16). Using (2.21) twice, we have

$$-i\omega G_{J^tJ^t}(\omega,\mathbf{k}) + i\mathbf{k}_iG_{J^iJ^t}(\omega,\mathbf{k}) = 0,$$
$$-i\omega G_{J^tJ^j}(\omega,\mathbf{k}) + i\mathbf{k}_iG_{J^iJ^j}(\omega,\mathbf{k}) = 0. \tag{2.22}$$

We next consider the time reversal invariance conditions (2.17) with $\{n_j\} = 0$. Since $\epsilon_{J^t} = +1$ and $\epsilon_{J^i} = -1$, we obtain

$$G_{J^iJ^t}(\omega,\mathbf{k}) = -G_{J^tJ^i}(\omega,-\mathbf{k}),$$
$$G_{J^iJ^j}(\omega,\mathbf{k}) = G_{J^jJ^i}(\omega,-\mathbf{k}). \tag{2.23}$$

Combing (A3) with (2.22) we therefore have the key result

$$\frac{1}{i\omega}\mathbf{k}_i\mathbf{k}_jG_{J^iJ^j}(\omega,\mathbf{k}) = -i\omega G_{J^tJ^t}(\omega,\mathbf{k}). \tag{2.24}$$

In general, taking the $\omega\to 0$ limit of the correlator $G_{AB}(\omega,\mathbf{k})$ gives rise to a static, thermodynamic susceptibility. It will be useful to write

$$-\lim_{\omega\to 0+i0}G_{J^tJ^t}(\omega,\mathbf{k}) \equiv \chi(\mathbf{k}), \tag{2.25}$$

where $\chi(\mathbf{k})$ is a charge-charge susceptibility (the sign here is explained in Appendix A). Note that (2.24) implies

$$\lim_{\omega\to 0+i0}\frac{1}{\omega^2}\mathbf{k}_i\mathbf{k}_jG_{J^iJ^j}(\omega,\mathbf{k}) = -\chi(\mathbf{k}), \tag{2.26}$$

and in particular, the longitudinal part of the current-current susceptibility vanishes, $\lim_{\omega\to 0+i0}\mathbf{k}_i\mathbf{k}_jG_{J^iJ^j}(\omega,\mathbf{k}) = 0$, provided that $\chi(\mathbf{k})$ is finite,[3] which we will assume.

In order to focus on studying the response to long-wavelength sources, it will be convenient to now rescale the wave-number $\mathbf{k}$ by ε and write (2.24) in the form

$$\frac{1}{i\omega}\mathbf{k}_i\mathbf{k}_jG_{J^iJ^j}(\omega,\varepsilon\mathbf{k}) = -\frac{i\omega}{\varepsilon^2}G_{J^tJ^t}(\omega,\varepsilon\mathbf{k}). \tag{2.27}$$

We next note that the ac conductivity matrix is defined by taking the following limit of the transport correlators:

$$\sigma^{ij}(\omega) = -\lim_{\varepsilon\to 0}\frac{1}{i\omega}G_{J^iJ^j}(\omega,\varepsilon\mathbf{k}). \tag{2.28}$$

Notice from the discussion above (2.16) that the real part of $\sigma^{ij}(\omega)$ is a positive semidefinite matrix. In general, the ac conductivity is a finite quantity for $\omega\neq 0$. On the other hand, the dc conductivity, defined by $\sigma_{dc}^{ij} \equiv \lim_{\omega\to 0}\sigma(\omega)$, is not necessarily finite. For example, if the system is translationally invariant or if the breaking of translation invariance has arisen spontaneously, or more generally if there are Goldstone modes present, generically the dc conductivity will be infinite, or more precisely there will be a delta function on the ac conductivity at $\omega = 0$. By taking the limit $\varepsilon\to 0$ in (2.27) we have

$$\mathbf{k}_i\mathbf{k}_j\sigma^{ij}(\omega) = i\omega\lim_{\varepsilon\to 0}\frac{1}{\varepsilon^2}G_{J^tJ^t}(\omega,\varepsilon\mathbf{k}). \tag{2.29}$$

Thus when the *dc* conductivity is finite, the function $G_{J^tJ^t}(\omega,\varepsilon\mathbf{k})/\varepsilon^2$ must have a pole at $\omega = 0$ after taking the $\varepsilon\to 0$ limit. Note that (2.25) shows that before the limit is taken this pole is absent (provided that $\chi(\mathbf{k})$ is finite).

[1] In this section we find it convenient to work with current vector densities. In Sec. III we will work with current vectors. We also note that as our analysis will focus on two-point functions of the current, we only require that the current to be conserved at the linearized level.

[2] We have also presented some more general results in Appendix A.

[3] Note that for a superfluid one can have $\chi(\mathbf{k})$ diverging at $\mathbf{k}\to 0$.

To make further progress, it is helpful to write

$$G_{J^tJ^t}(\omega,\varepsilon\mathbf{k})\chi(\varepsilon\mathbf{k})^{-1}=\frac{-N(\omega,\varepsilon\mathbf{k})}{-i\omega+N(\omega,\varepsilon\mathbf{k})}, \tag{2.30}$$

where we have defined the quantity

$$N(\omega,\varepsilon\mathbf{k})=\frac{G_{J^tJ^t}(\omega,\varepsilon\mathbf{k})}{\frac{1}{i\omega}(G_{J^tJ^t}(\omega,\varepsilon\mathbf{k})+\chi(\varepsilon\mathbf{k}))}. \tag{2.31}$$

We can now prove that $N(\omega,\varepsilon\mathbf{k})$ is an analytic function of ω provided that $\mathrm{Im}(\omega)\neq 0$. First, any poles in the numerator $G_{J^tJ^t}(\omega,\varepsilon\mathbf{k})$, which can only occur in the lower half plane, will cancel out with those in the denominator. We thus need to check whether or not the denominator in (2.31) can vanish for $\mathrm{Im}(\omega)\neq 0$. That this cannot occur can be seen by writing

$$\frac{1}{i\omega}(G_{J^tJ^t}(\omega,\varepsilon\mathbf{k})+\chi(\varepsilon\mathbf{k}))=\int_{C_1}\frac{d\omega'}{i\pi}\frac{ImG_{J^tJ^t}(\omega',\varepsilon\mathbf{k})}{\omega'(\omega'-\omega)}, \tag{2.32}$$

where C_1 is a contour that skirts just under the real axis. Then writing $\omega=x+iy$, with $y\neq 0$, we can show that the real part of the integral is nonvanishing after using the fact that $\mathrm{Im}G_{J^tJ^t}(\omega',\varepsilon\mathbf{k})/\omega'\leq 0$, which we showed in (2.16). We now return to (2.29), from which we deduce that, for fixed ω, as $\varepsilon\to 0$, we can expand

$$N=\varepsilon^2\frac{\mathbf{k}_i\mathbf{k}_j\sigma^{ij}(\omega)}{\chi(0)}+\cdots, \tag{2.33}$$

with the neglected terms going to zero with a higher power of ε.

We are now in a position to discuss the poles of $G_{J^tJ^t}(\omega,\varepsilon\mathbf{k})$ that appear at the "origin," by which we mean when both $\omega\to 0$ and $\varepsilon\to 0$. The simplest possibility is if $N(\omega,\varepsilon\mathbf{k})$ does not have any poles (or branch cuts) at $\omega=0$. In this case, we see that when the dc conductivity matrix is finite, $G_{J^tJ^t}$ will have a single diffusion pole with dispersion relation

$$\omega=-i\varepsilon^2D(k)+\cdots,\qquad D(k)=[\sigma^{ij}_{dc}k_ik_j]\chi(\mathbf{0})^{-1}, \tag{2.34}$$

and the neglected terms are higher order in ε. This is our first result on Einstein relations for inhomogeneous media.

It is important to emphasize that is not the only possibility. Indeed, as we discuss in the next subsection, there are additional poles when there are additional conserved currents. If, for example, we suppose that there are two conserved currents in total then a second diffusion pole can appear in $G_{J^tJ^t}(\omega,\varepsilon\mathbf{k})$. To illustrate this situation schematically, consider the behavior of the following function for $\omega,\varepsilon\mathbf{k}\to 0$:

$$\varepsilon^2\left(\frac{A}{-i\omega+\varepsilon^2a}+\frac{B}{-i\omega+\varepsilon^2b}\right)\sim\frac{\varepsilon^2(A+B)}{-i\omega+\varepsilon^2(\frac{aA+bB}{A+B}-i\frac{AB(a-b)^2}{(A+B)^2}\frac{\varepsilon^2}{\omega}+\mathcal{O}(\frac{\varepsilon^2}{\omega})^2)}, \tag{2.35}$$

corresponding to the function $N(\omega,\varepsilon\mathbf{k})$ having additional singularities at $\omega\to 0$. Another interesting situation in which additional poles will appear is in the presence of Goldstone modes arising from broken symmetries. Additional general statements can be made using the memory matrix formalism, generalizing the discussion in [1].

Returning now to the case in which there is just a single conserved current with a single diffusion pole then a natural phenomenological expression for the Green's function is given near the origin, $(\omega,\varepsilon\mathbf{k})\to 0$, by

$$G_{J^tJ^t}(\omega,\varepsilon\mathbf{k})\sim\frac{-D(\omega,\varepsilon\mathbf{k})}{-i\omega+D(\omega,\varepsilon\mathbf{k})}\chi(\varepsilon\mathbf{k}), \tag{2.36}$$

with $D(\omega,\varepsilon\mathbf{k})\sim\varepsilon^2\frac{\mathbf{k}_i\mathbf{k}_j\sigma^{ij}(\omega)}{\chi(\varepsilon\mathbf{k})}$. It is interesting to note that if, by contrast, we are in the context of infinite dc conductivity with $\sigma^{ij}(\omega)\sim K^{ij}(\frac{i}{\omega}+\pi\delta(\omega))$ for small ω, where K^{ij} is constant, then (2.36) is gives rise to sound modes for the current density J^t, with dispersion relation $\omega_\pm=\pm\varepsilon\sqrt{K^{ij}\mathbf{k}_i\mathbf{k}_j/\chi(\varepsilon\mathbf{k})}$. The transition between diffusion modes and sound modes was also discussed in a homogeneous hydrodynamic setting, with a phenomenological term to relax momentum, in [6].

To conclude this subsection, we briefly note that we can carry out a similar analysis for the higher Fourier modes of the current-current correlators. Starting with (2.21), the analogue of (2.27) is

$$\frac{1}{i\omega}(\mathbf{k}+n_r\mathbf{k}_L^r)_i\mathbf{k}_jG^{(\{n_l\})}_{J^i_AJ^j_B}(\omega,\mathbf{k})=-i\omega G^{(\{n_l\})}_{J^t_AJ^t_B}(\omega,\mathbf{k}), \tag{2.37}$$

and this leads, *mutatis-mutandis*, to additional relations concerning the poles of $G^{(\{n_l\})}_{J^t_AJ^t_B}(\omega,\mathbf{k})$, which would be interesting to explore in more detail. We note however, that for $\{n_l\}\neq 0$, there is no longer a simple statement concerning the positivity of $\mathrm{Im}G^{(\{n_l\})}_{J^t_AJ^t_B}(\omega,\mathbf{k})/\omega$, which was used in the above. We also point out that within a holographic context and for a specific gravitational model, some of the $G^{(\{n_l\})}_{J^i_AJ^j_B}(\omega,\mathbf{k})$ were calculated in [28].

B. Generalized Einstein relations for multiple currents

We now assume that we have multiple conserved currents J^μ_A. For example, one could have both a conserved heat current and a conserved $U(1)$ current. Much of the analysis that we carried out for the case of a single current goes through straightforwardly and we obtain

$$\frac{1}{i\omega}\mathbf{k}_i\mathbf{k}_j G_{J_A^i J_B^j}(\omega, \varepsilon\mathbf{k}) = -\frac{i\omega}{\varepsilon^2} G_{J_A^t J_B^t}(\omega, \varepsilon\mathbf{k}). \tag{2.38}$$

We write the charge susceptibilities and the ac conductivity via

$$\begin{aligned}\chi_{AB}(\varepsilon\mathbf{k}) &= -\lim_{\omega\to 0+i0} G_{J_A^t J_B^t}(\omega, \varepsilon\mathbf{k}),\\ \sigma_{AB}^{ij}(\omega) &= -\lim_{\varepsilon\to 0}\frac{1}{i\omega} G_{J_A^i J_B^j}(\omega, \varepsilon\mathbf{k}),\end{aligned} \tag{2.39}$$

respectively, and we now have

$$\mathbf{k}_i\mathbf{k}_j\sigma_{AB}^{ij}(\omega) = i\omega\lim_{\varepsilon\to 0}\frac{1}{\varepsilon^2} G_{J_A^t J_B^t}(\omega, \varepsilon\mathbf{k}). \tag{2.40}$$

Generically this shows that for finite dc conductivities there will be at least as many poles in the transport current correlators as there are currents.

Proceeding much as before we write

$$\mathbf{G}(\omega, \varepsilon\mathbf{k})\boldsymbol{\chi}(\varepsilon\mathbf{k})^{-1} = -[-i\omega + \mathbf{N}(\omega, \varepsilon\mathbf{k})]^{-1}\mathbf{N}(\omega, \varepsilon\mathbf{k}), \tag{2.41}$$

where $\mathbf{G}(\omega, \varepsilon\mathbf{k})_{AB} \equiv G_{J_A^t J_B^t}(\omega, \varepsilon\mathbf{k})$ and

$$\mathbf{N}(\omega, \varepsilon\mathbf{k}) \equiv \mathbf{G}(\omega, \varepsilon\mathbf{k})\left[\frac{1}{i\omega}(\mathbf{G}(\omega, \varepsilon\mathbf{k}) + \boldsymbol{\chi}(\varepsilon\mathbf{k}))\right]^{-1}. \tag{2.42}$$

We can again argue that $\mathbf{N}(\omega, \varepsilon\mathbf{k})$ can only have poles on the real ω axis. From (2.40) we deduce that for fixed ω, as $\varepsilon \to 0$, we can expand

$$\mathbf{N}(\omega, \varepsilon\mathbf{k}) = \varepsilon^2\Sigma(\omega, \mathbf{k})\boldsymbol{\chi}(\varepsilon\mathbf{k})^{-1}, \tag{2.43}$$

where $\Sigma(\omega, \mathbf{k})_{AB} = \mathbf{k}_i\mathbf{k}_j\sigma_{AB}^{ij}(\omega)$ and the neglected terms go to zero with a higher power of ε.

If we now assume that $\mathbf{N}(\omega, \varepsilon\mathbf{k})$ doesn't have any poles at $\omega = 0$, then we can conclude that at the origin, i.e. when both $\omega \to 0$ and $\varepsilon \to 0$, if the dc conductivities are finite then the diffusion poles of the system are located at

$$\omega_A(\mathbf{k}) = -iD_A(\mathbf{k})\varepsilon^2 + \cdots, \tag{2.44}$$

where $D_A(\mathbf{k})$ are the eigenvalues of what can be called the "generalized diffusion matrix" $\mathbf{D}(\mathbf{k})$ defined by

$$\mathbf{D}(\mathbf{k}) = \Sigma(0, \mathbf{k})\boldsymbol{\chi}(\mathbf{0})^{-1}, \tag{2.45}$$

and the dots involve higher order corrections in ε. In particular when the dc conductivities are finite, the number of diffusion poles is the same as the number of conserved currents.

Furthermore, we emphasize that when there is more than one conserved current, generically, these diffusion modes do not satisfy a dispersion relation of the form $\omega \sim -i\varepsilon^2\Sigma_{ij}k^ik^j$, with the matrix Σ_{ij} a component of the dc conductivities. As a consequence we refer to our result (2.44), (2.45) as a "generalized Einstein relation."

We conclude this section by noting that the general result (2.44), (2.45) relates thermodynamic instabilities to dynamic instabilities. Suppose that the system has a static susceptibility matrix $\boldsymbol{\chi}(\mathbf{0})$ with a negative eigenvalue and hence is thermodynamically unstable. Then (2.45) implies that $\mathbf{D}(\mathbf{k})$ will have a negative eigenvalue, for small $\mathbf{k}$, and hence, from (2.44) we deduce that there will be a diffusion pole in the upper half plane leading to a dynamical instability.[4]

III. DIFFUSION IN RELATIVISTIC HYDRODYNAMICS

We now discuss thermoelectric transport within the context of relativistic hydrodynamics. As well as generalizing the work of [14] to include a conserved $U(1)$ charge (as also studied in [18]), we will also be able to use the formalism to illustrate the results of the previous section. In particular, associated with the heat current and the $U(1)$ current we construct two diffusion modes with dispersion relations satisfying the generalized Einstein relation (2.44). We note that it will be convenient to use a slightly different notation in this section, which implies that a little care is required in directly comparing with the last section.

A. General setup

We will consider an arbitrary relativistic quantum field theory with a global $U(1)$ symmetry in $d \geq 2$ spacetime dimensions. The field theory is defined on a static, curved manifold, with metric $g_{\mu\nu}$, and a nonzero background gauge field, A_μ, of the form:

$$\begin{aligned}ds^2 &= -f^2(x)dt^2 + h_{ij}(x)dx^idx^j,\\ A_t &= a_t(x).\end{aligned} \tag{3.1}$$

This corresponds to studying the field theory with f^2 and h_{ij} parametrizing sources for the stress tensor components T^{tt} and T^{ij}, respectively, and a_t parametrizing a source for the J^t component of the conserved $U(1)$ current. We focus on cases in which the manifold has planar topology, with the globally defined spatial coordinates x^i parametrizing $\mathbb{R}^{d-1}$, and f, h_{ij},a_t all depending periodically on x^i, with period L_i.

[4]An explicit example of such a dynamic instability can be seen using the results of Appendix B.

We will study the field theory at finite temperature in the hydrodynamic limit keeping the leading order viscous terms. In particular, we will consider temperatures[5] that are much greater than the largest wave number that appears in the background fields in (3.1). The Ward identities are given by

$$D_\mu T^{\mu\nu} = F^{\nu\lambda} J_\lambda, \qquad D_\mu J^\mu = 0, \tag{3.2}$$

where D_μ is the covariant derivative with respect to $g_{\mu\nu}$ and $F_{\mu\nu} = 2\partial_{[\mu} A_{\nu]}$. For the special case of conformal field theory, we should also impose $T^\mu{}_\mu = 0$ and this implies, amongst other things, that in (3.3) $\zeta_b = 0$ and $\epsilon = (d-1)P$.

The hydrodynamic variables are the local temperature, $T(x)$, the local chemical potential, $\mu(x)$, and the fluid velocity, u^μ, with $u^\mu u^\nu g_{\mu\nu} = -1$. As in [29], the constitutive relations are given, in the Landau frame, by[6]

$$\begin{aligned} T_{\mu\nu} &= P g_{\mu\nu} + (P+\epsilon) u_\mu u_\nu \\ &\quad - 2\eta\left(D_{(\mu}u_{\nu)} + u_\rho u_{(\mu} D^\rho u_{\nu)} - (g_{\mu\nu} + u_\mu u_\nu)\frac{D_\rho u^\rho}{d-1}\right) \\ &\quad - \zeta_b (g_{\mu\nu} + u_\mu u_\nu) D_\rho u^\rho, \\ J^\mu &= \rho u^\mu + \sigma_Q \left(F^{\mu\nu} u_\nu - T(g^{\mu\nu} + u^\mu u^\nu) D_\nu\left(\frac{\mu}{T}\right)\right), \end{aligned} \tag{3.3}$$

where P is the pressure density, ϵ is the energy density and ρ is the $U(1)$ charge density. The dissipative terms in (3.3) are the shear viscosity, η, the bulk viscosity, ζ_b and the conductivity, σ_Q, which should not be confused with the electrical dc conductivity, σ_{dc}, which we discuss later. We also have the local thermodynamic relation and first law which take the form

$$P + \epsilon = sT + \mu\rho, \qquad dP = s dT + \rho d\mu, \tag{3.4}$$

where s is the entropy density. It will also be helpful to introduce the susceptibilities c_μ, ξ and χ via

$$ds = T^{-1} c_\mu dT + \xi d\mu, \qquad d\rho = \xi dT + \chi d\mu. \tag{3.5}$$

For any vector k, the Ward identities imply

$$D_\mu[(T^\mu{}_\nu + J^\mu A_\nu) k^\nu] = \frac{1}{2}\mathcal{L}_k g_{\mu\nu} T^{\mu\nu} + \mathcal{L}_k A_\mu J^\mu, \tag{3.6}$$

where $\mathcal{L}_k$ is the Lie derivative. Taking $k = \partial_t$ we define the heat current as

$$Q^\mu = -(T^\mu{}_t + A_t J^\mu), \tag{3.7}$$

which is conserved for stationary metrics with $\mathcal{L}_k A_\nu = 0$. Thus, given such background metrics and gauge fields, for time-independent configurations we therefore have $\partial_i(\sqrt{-g} Q^i) = \partial_i(\sqrt{-g} J^i) = 0$.

In thermal equilibrium the fluid configuration is given by

$$u_t = -f(x), \qquad u_i = 0, \qquad T = T_0(x), \qquad \mu = \mu_0(x), \tag{3.8}$$

where $T_0(x)$ and $\mu_0(x)$ are periodic functions, and from (3.4) we have the equilibrium relations

$$P_0 + \epsilon_0 = s_0 T_0 + \mu_0 \rho_0, \qquad \partial_i P_0 = s_0 \partial_i T_0 + \rho_0 \partial_i \mu_0. \tag{3.9}$$

For later use, we note that we also have

$$\begin{aligned} \nabla_i s_0 &= T_0^{-1} c_{\mu 0} \nabla_i T_0 + \xi_0 \nabla_i \mu_0, \\ \nabla_i \rho_0 &= \xi_0 \nabla_i T_0 + \chi_0 \nabla_i \mu_0. \end{aligned} \tag{3.10}$$

By calculating $T^{\mu\nu}$, J^μ one can show that the Ward identities are satisfied provided that

$$T_0 = f^{-1} \bar{T}_0, \qquad \mu_0 = f^{-1} a_t, \tag{3.11}$$

where $\bar{T}_0$ is constant. Note, in particular, that in thermal equilibrium the local hydrodynamic variable T_0 is not constant when f is not constant and, furthermore, there is a factor of f that appears in the relationship between μ_0 and the background gauge field. We also note that we have set a possible integration constant to zero in the second expression as we want μ_0 to vanish when a_t does. Finally it will be helpful to define the zero mode of a_t via $\bar{\mu}_0 \equiv \oint a_t$, where we are again using the notation $\oint \equiv (L_1 \cdots L_d)^{-1} \int_{\{0\}}^{\{L_i\}} dx^1 \cdots dx^d$. This allows us to write $\mu_0 = f^{-1}(\bar{\mu}_0 + \tilde{a}_t(x))$, with $\oint \tilde{a}_t = 0$.

The nonvanishing components of the stress tensor and current for this equilibrium configuration are then given by

$$T_{tt} = \epsilon_0 f^2, \qquad T_{ij} = P_0 h_{ij}, \qquad J^t = \rho_0 f^{-1}. \tag{3.12}$$

In particular for the backgrounds we are considering, in thermal equilibrium both the electric and the heat currents vanish: $J^i = Q^i = 0$. Note, since (3.1) provides a source for the energy and the charge, we can immediately deduce that the charge-current susceptibilities must vanish. The total energy and charge of the equilibrium configuration are defined by

[5] This temperature is the same as what is denoted as $\bar{T}_0$ below.

[6] Following [29], we have set to zero two other terms in J^μ that are allowed by Lorentz invariance but are not consistent with positivity of entropy and thermodynamics with external sources.

$$\epsilon_{\rm tot} = -\oint \sqrt{-g}T^t{}_t = \oint \sqrt{h} f \epsilon_0,$$
$$\rho_{\rm tot} = \oint \sqrt{-g}J^t = \oint \sqrt{h}\rho_0. \tag{3.13}$$

We can also define the total equilibrium entropy as

$$s_{\rm tot} = \oint \sqrt{h} s_0. \tag{3.14}$$

For later use, using the fact that s_0 is a function of T_0 and μ_0, we observe that for suitable zero modes of the charge susceptibilities we have

$$\frac{\partial s_{\rm tot}}{\partial \bar{T}_0} = \oint \sqrt{h} f^{-1} T_0^{-1} c_{\mu 0},$$
$$\frac{\partial s_{\rm tot}}{\partial \bar{\mu}_0} = \oint \sqrt{h} f^{-1} \xi_0. \tag{3.15}$$

Similarly, we also have

$$\frac{\partial \rho_{\rm tot}}{\partial \bar{T}_0} = \oint \sqrt{h} f^{-1} \xi_0, \qquad \frac{\partial \rho_{\rm tot}}{\partial \bar{\mu}_0} = \oint \sqrt{h} f^{-1} \chi_0. \tag{3.16}$$

B. Generalized Navier-Stokes equations

In the following we want to study the behavior of small perturbations about the equilibrium configuration, including the possibility of adding external, perturbative thermal gradient and electric field sources. Following [14] we will do this by considering

$$ds^2 = -f^2(1-2\phi_T)dt^2 + h_{ij}dx^i dx^j,$$
$$A_t = a_t - f\mu_0\phi_T + \phi_E, \tag{3.17}$$

along with

$$u_t = -f(1-\phi_T), \qquad u_i = \delta u_i,$$
$$T = T_0 + \delta T, \qquad \mu = \mu_0 + \delta\mu. \tag{3.18}$$

Here $\phi_T, \phi_E, \delta u_i, \delta T$ and $\delta\mu$ are all functions of (t, x^i). Note that these need not be periodic functions of the spatial coordinates. For later use, we also define the spatial components of the external sources ζ_i, E_i via

$$\zeta_i = \partial_i \phi_T, \qquad E_i = \partial_i \phi_E. \tag{3.19}$$

At linearized order, the perturbed stress tensor and $U(1)$ current can then be written as

$$T_{tt} = \epsilon_0 f^2(1-2\phi_T) + \delta\epsilon f^2,$$
$$T_{ti} = -f(P_0+\epsilon_0)\delta u_i,$$
$$T_{ij} = (P_0+\delta P)h_{ij} - 2\eta_0 f^{-1}\left(\nabla_{(i}(f\delta u_{j)}) - \frac{h_{ij}}{(d-1)}\nabla_k(f\delta u^k)\right) - \zeta_{b0} h_{ij} f^{-1}\nabla_k(f\delta u^k),$$
$$J^t = \rho_0 f^{-1}(1+\phi_T) + f^{-1}\delta\rho,$$
$$J^i = \rho_0\delta u^i + \sigma_{Q0} f^{-1}[E^i - \nabla^i(f\delta\mu) - f\mu_0\zeta^i + \mu_0 T_0^{-1}\nabla^i(f\delta T)], \tag{3.20}$$

where ∇_i is the covariant derivative with respect to the metric h_{ij}, which is also used to raise and lower indices. The Ward identities (3.2) give

$$\partial_t\delta\rho + \nabla_i(fJ^i) = 0,$$
$$f\partial_t\delta\epsilon + \nabla_i(f^2(P_0+\epsilon_0)\delta u^i) - fJ^i\nabla_i a_t = 0,$$
$$f^{-1}(P_0+\epsilon_0)\partial_t\delta u_j - 2f^{-1}\nabla^i(\eta_0\nabla_{(i}(f\delta u_{j)})) + f^{-1}\nabla_j\left(\left(\frac{2\eta_0}{(d-1)} - \zeta_{b0}\right)\nabla_k(f\delta u^k)\right)$$
$$= -\nabla_j\delta P - (\delta\epsilon+\delta P)f^{-1}\nabla_j f + (P_0+\epsilon_0)\zeta_j + \rho_0(f^{-1}E_j - \mu_0\zeta_j) + f^{-1}\delta\rho\nabla_j a_t. \tag{3.21}$$

In the case when there is no $U(1)$ charge this agrees with the expression derived in Eqs. (A.10) of [14]. These expressions can be further simplified. We use (3.11) as well as

$$\delta P = s_0\delta T + \rho_0\delta\mu, \qquad \delta\epsilon = T_0\delta s + \mu_0\delta\rho,$$
$$\delta s = T_0^{-1}c_{\mu 0}\delta T + \xi_0\delta\mu, \qquad \delta\rho = \xi_0\delta T + \chi_0\delta\mu, \tag{3.22}$$

which we obtain from (3.4), (3.5). After also using (3.10) we eventually find that we can rewrite the system (3.21) in the following form, which is the key result of this section,

$$
\begin{aligned}
&\xi_0\partial_t\delta T+\chi_0\partial_t\delta\mu+\nabla_i(fJ^i)=0,\\
&\quad fc_{\mu 0}\partial_t\delta T+fT_0\xi_0\partial_t\delta\mu+\nabla_i(fQ^i)=0,\\
&\quad (P_0+\epsilon_0)\partial_t\delta u_j-2\nabla^i(\eta_0\nabla_{(i}(f\delta u_{j)}))\\
&\quad +\nabla_j\left(\left(\frac{2\eta_0}{(d-1)}-\zeta_{b0}\right)\nabla_k(f\delta u^k)\right)\\
&=\rho_0[E_j-\nabla_j(f\delta\mu)]+fT_0s_0[\zeta_j-(fT_0)^{-1}\nabla_j(f\delta T)],
\end{aligned}
\tag{3.23}
$$

with

$$
\begin{aligned}
J^i&=\rho_0\delta u^i+\sigma_{Q0}f^{-1}[E^i-\nabla^i(f\delta\mu)]\\
&\quad-\sigma_{Q0}\mu_0[\zeta^i-(fT_0)^{-1}\nabla^i(f\delta T)],\\
Q^i&=f(P_0+\epsilon_0)\delta u^i-f\mu_0J^i.
\end{aligned}
\tag{3.24}
$$

Notice that the first two lines in (3.23) are just current conservation equations for the linearized perturbation. We emphasize that all background equilibrium quantities, marked with a 0 subscript, are all periodic functions of the spatial coordinates. It is interesting to note that the system of Eqs. (3.23) is invariant under the interchange

$$
E_j\leftrightarrow-\nabla_j(f\delta\mu),\qquad \zeta_j\leftrightarrow-f^{-1}T_0^{-1}\nabla_j(f\delta T).
\tag{3.25}
$$

Finally, for later use, we note that when the sources are set to zero, $\phi_T=\phi_E=0$, we have for the total charges

$$
\begin{aligned}
&\oint\sqrt{-g}J^t=\oint\sqrt{h}\rho_0+\oint\sqrt{h}\delta\rho,\\
&\oint\sqrt{-g}Q^t=\oint\sqrt{h}f(\epsilon_0-\mu_0\rho_0)+\bar{T}_0\oint\sqrt{h}\delta s.
\end{aligned}
\tag{3.26}
$$

C. Thermoelectric dc conductivity

We now explain how we can obtain the thermoelectric dc conductivity, generalizing [14]. We begin by considering the sources ϕ_T and ϕ_E to have space and time dependence of the form $e^{-i\omega t}e^{ik_ix^i}$, where k_i is an arbitrary wave number. After solving (3.23) for δu_j, $\delta\mu$, δT one obtains the local currents J^i Q^i, and hence the current fluxes $\bar{J}^i$ $\bar{Q}^i$, as functions of E_i and ζ_i. To obtain the thermoelectric dc conductivity we should then take the limit $k_i\to 0$, followed by $\omega\to 0$.

By considering approximating $e^{ik_ix^i}\sim 1+ik_ix^i$ we are prompted[7] to consider a time-independent source of the form

$$
\phi_T=x^i\bar{\zeta}_i,\qquad \phi_E=x^i\bar{E}_i,
\tag{3.27}
$$

[7] An alternative procedure is to consider sources that are linear in time, as explained in a holographic context in [30,31].

where $\bar{\zeta}_i$, $\bar{E}_i$ are constants and hence $E_i=\bar{E}_i$, $\zeta_i=\bar{\zeta}_i$. After substituting into (3.32) we obtain the system[8]

$$
\begin{aligned}
&\nabla_i(fJ^i)=0,\qquad \nabla_i(fQ^i)=0,\\
&-2\nabla^i(\eta_0\nabla_{(i}(f\delta u_{j)}))+\nabla_j\left(\left(\frac{2\eta_0}{(d-1)}-\zeta_{b0}\right)\nabla_k(f\delta u^k)\right)\\
&=\rho_0\bar{E}_j-\rho_0\nabla_j(f\delta\mu)+fs_0T_0\bar{\zeta}_j-s_0\nabla_j(f\delta T).
\end{aligned}
\tag{3.28}
$$

After solving these equations we obtain the local time-independent, steady state currents $J^i(x)Q^i(x)$, periodic in the spatial coordinate, as functions of $\bar{\zeta}_i$, $\bar{E}_i$. We can now define the heat and charge current fluxes via

$$
\begin{aligned}
\bar{Q}^i&\equiv\oint\sqrt{-g}Q^i=\oint\sqrt{h}fQ^i,\\
\bar{J}^i&\equiv\oint\sqrt{-g}J^i=\oint\sqrt{h}fJ^i,
\end{aligned}
\tag{3.29}
$$

and the dc conductivities are obtained from

$$
\begin{pmatrix}\bar{J}^i\\ \bar{Q}^i\end{pmatrix}=\begin{pmatrix}\sigma_{dc}^{ij} & \bar{T}_0\alpha_{dc}^{ij}\\ \bar{T}_0\bar{\alpha}_{dc}^{ij} & \bar{T}_0\bar{\kappa}_{dc}^{ij}\end{pmatrix}\begin{pmatrix}\bar{E}_j\\ \bar{\zeta}_j\end{pmatrix}.
\tag{3.30}
$$

Since we are considering backgrounds which preserve time reversal invariance the Onsager relations imply that σ_{dc} and $\bar{\kappa}_{dc}$ are symmetric matrices and $\alpha_{dc}^T=\bar{\alpha}_{dc}$.

D. Diffusive modes

We now discuss how we can construct a perturbative diffusive solution of the system of Eqs. (3.23) that is associated with diffusion modes. Our objective will be to extract the associated dispersion relations for these modes.

We first set the source terms in (3.23) to zero: $E_i=\zeta_i=0$. We will allow for a time dependence of the form $e^{-i\omega t}$ and consider the expansion

$$
\omega=\sum_{\alpha=1}^{\infty}\varepsilon^{\alpha}\omega^{(\alpha)},
\tag{3.31}
$$

with $\varepsilon\ll 1$. Since we are interested in wavelengths that are much larger than the periods, L_i, of the background fields in (3.1), we introduce arbitrary wave numbers k^i and consider

[8] In the special case of conformal field theories, similar equations were obtained in a holographic context in [32]. The equations differ when there is a $U(1)$ symmetry due to a difference in the expression for Q^i in (3.24). The equations should agree in the hydrodynamic limit, after a possible change of frame and/or incorporating higher order terms in the hydrodynamic expansion, and it would be interesting to investigate this in more detail.

$$\delta T = e^{-i\omega t}e^{i\epsilon k_i x^i}\sum_{\alpha=0}^{\infty}\varepsilon^{\alpha}\delta T^{(\alpha)}(x),$$
$$\delta\mu = e^{-i\omega t}e^{i\epsilon k_i x^i}\sum_{\alpha=0}^{\infty}\varepsilon^{\alpha}\delta\mu^{(\alpha)}(x),$$
$$\delta u_i = e^{-i\omega t}e^{i\epsilon k_i x^i}\sum_{\alpha=0}^{\infty}\varepsilon^{\alpha}\delta u_i^{(\alpha)}(x), \tag{3.32}$$

with the functions inside the summations taken to be periodic in the x^i, with period L^i.

We next note that the system of Eqs. (3.23) (with $E^i = \zeta^i = 0$) admit the simple time-independent solution with $f\delta T$, $f\delta\mu$ both constant and $\delta u_i = 0$. Indeed, from (3.11) this corresponds to simply perturbing the parameters of the thermal equilibrium configuration. The diffusive modes are constructed as a perturbation of this time-independent solution by using the expansions (3.31), (3.32) and taking

$$f\delta T^{(0)} = \text{constant}, \qquad f\delta\mu^{(0)} = \text{constant}, \qquad \delta u_i^{(0)} = 0, \tag{3.33}$$

as the zeroth order solution. We immediately see that the associated expansion for J^i and Q^i can be written as

$$J^i = e^{-i\omega t}e^{i\epsilon k_i x^i}\sum_{\alpha=1}^{\infty}\varepsilon^{\alpha}J^{i(\alpha)}(x),$$
$$Q^i = e^{-i\omega t}e^{i\epsilon k_i x^i}\sum_{\alpha=1}^{\infty}\varepsilon^{\alpha}Q^{i(\alpha)}(x). \tag{3.34}$$

At leading order in ε, the first two equations of (3.23) then read

$$-i\omega^{(1)}\xi_0\delta T^{(0)} - i\omega^{(1)}\chi_0\delta\mu^{(0)} + \nabla_i(fJ^{i(1)}) = 0,$$
$$-i\omega^{(1)}c_{\mu 0}f\delta T^{(0)} - i\omega^{(1)}T_0\xi_0 f\delta\mu^{(0)} + \nabla_i(fQ^{i(1)}) = 0. \tag{3.35}$$

Integrating Eq. (3.35) over a period we obtain

$$i\omega^{(1)}\oint\sqrt{h}(\xi_0\delta T^{(0)} + \chi_0\delta\mu^{(0)}) = 0,$$
$$i\omega^{(1)}\oint\sqrt{h}f(c_{\mu 0}\delta T^{(0)} + T_0\xi_0\delta\mu^{(0)}) = 0. \tag{3.36}$$

Assuming thermodynamically stable matter, the matrix of static susceptibilities, whose components appear in (3.36), is positive definite and these equations can only be satisfied by setting $\omega^{(1)} = 0$. The leading order system (3.23) then becomes

$$\nabla_i(fJ^{i(1)}) = 0, \qquad \nabla_i(fQ^{i(1)}) = 0,$$
$$-2\nabla^i(\eta_0\nabla_{(i}(f\delta u_{j)}^{(1)}))$$
$$+\nabla_j\left(\left(\frac{2\eta_0}{(d-1)} - \zeta_{b0}\right)\nabla_k(f\delta u^{k(1)})\right)$$
$$= -i\rho_0 k_j f\delta\mu^{(0)} - \rho_0\nabla_j(f\delta\mu^{(1)}) - is_0 k_j f\delta T^{(0)}$$
$$- s_0\nabla_j(f\delta T^{(1)}), \tag{3.37}$$

with

$$J^{i(1)} = \rho_0\delta u^{i(1)} + \sigma_{Q0}f^{-1}[-\nabla^i(f\delta\mu^{(1)})]$$
$$- \sigma_{Q0}\mu_0[-(fT_0)^{-1}\nabla^i(f\delta T^{(1)})],$$
$$Q^{i(1)} = f(P_0 + \epsilon_0)\delta u^{i(1)} - f\mu_0 J^{i(1)}. \tag{3.38}$$

Notice that this system is equivalent to the system of Eqs. (3.28) that appeared for the calculation of the thermoelectric dc conductivity if we identify $\bar{E}_i \leftrightarrow -ik_i f\delta\mu^{(0)}$, $\bar{\zeta}_i \leftrightarrow -ik_j T_0^{-1}\delta T^{(0)}$ and note that the quantities on the right-hand sides of these expressions are indeed constant. Thus, we can express the heat current fluxes $\bar{J}^{i(1)}$ and $\bar{Q}^{i(1)}$ in terms of $-ik_i f\delta\mu^{(0)}$, $-ik_j T_0^{-1}\delta T^{(0)}$ using the thermoelectric dc conductivity matrix given in (3.30) to get

$$\bar{J}^{i(1)} \equiv \oint\sqrt{h}fJ^{i(1)} = -i\sigma_{dc}^{ij}k_j f\delta\mu^{(0)} - i\alpha_{dc}^{ij}k_j f\delta T^{(0)},$$
$$\bar{Q}^{i(1)} \equiv \oint\sqrt{h}fQ^{i(1)} = -i\bar{T}_0\alpha_{dc}^{ij}k_j f\delta\mu^{(0)} - i\bar{\kappa}_{dc}^{ij}k_j f\delta T^{(0)}. \tag{3.39}$$

Continuing the expansion, we next examine the first two equations of (3.23) at second order in ε to find

$$-i\omega^{(2)}\xi_0\delta T^{(0)} - i\omega^{(2)}\chi_0\delta\mu^{(0)} + ik_i fJ^{i(1)} + \nabla_i(fJ^{i(2)}) = 0,$$
$$-i\omega^{(2)}c_{\mu 0}f\delta T^{(0)} - i\omega^{(2)}T_0\xi_0 f\delta\mu^{(0)} + ik_i fQ^{i(1)}$$
$$+\nabla_i(fQ^{i(2)}) = 0. \tag{3.40}$$

Integrating these two equations over a period, substituting the expression for the dc conductivity and using (3.15), (3.16) we now deduce

$$i\omega^{(2)}\left(\frac{\partial\rho_{\text{tot}}}{\partial\bar{T}_0}f\delta T^{(0)} + \frac{\partial\rho_{\text{tot}}}{\partial\bar{\mu}_0}f\delta\mu^{(0)}\right)$$
$$-\alpha_{dc}^{ij}k_ik_jf\delta T^{(0)} - \sigma_{dc}^{ij}k_ik_jf\delta\mu^{(0)} = 0,$$
$$i\omega^{(2)}\bar{T}_0\left(\frac{\partial s_{\text{tot}}}{\partial\bar{T}_0}f\delta T^{(0)} + \frac{\partial s_{\text{tot}}}{\partial\bar{\mu}_0}f\delta\mu^{(0)}\right)$$
$$-\bar{\kappa}_{dc}^{ij}k_ik_jf\delta T^{(0)} - \bar{T}_0\alpha_{dc}^{ij}k_ik_jf\delta\mu^{(0)} = 0. \tag{3.41}$$

Writing this in matrix form as

$$\mathbb{M}\begin{pmatrix} f\delta T^{(0)} \\ f\delta\mu^{(0)} \end{pmatrix} = 0, \tag{3.42}$$

we have $\det(\mathbb{M}) = 0$. This gives rises to a quadratic equation for $i\omega^{(2)}$ which has two solutions, $i\omega_{\pm}^{(2)}$, which give the leading order dispersion relations for the diffusion modes that we are after.

To write $i\omega_{\pm}^{(2)}$ in a compact way we first define the scalar quantities depending on the dc conductivities that are quadratic in the wave numbers k^i:

$$\bar{\kappa}(k) \equiv \bar{\kappa}_{dc}^{ij} k_i k_j, \qquad \alpha(k) \equiv \alpha_{dc}^{ij} k_i k_j, \qquad \sigma(k) \equiv \sigma_{dc}^{ij} k_i k_j, \tag{3.43}$$

as well as

$$\kappa(k) \equiv \bar{\kappa}(k) - \frac{\alpha(k)^2 \bar{T}_0}{\sigma(k)}. \tag{3.44}$$

Recall that $\kappa_{dc}^{ij} \equiv \bar{\kappa}_{dc}^{ij} - \bar{T}_0(\bar{\alpha}_{dc} \cdot \sigma_{dc}^{-1} \cdot \alpha_{dc})^{ij}$ is the dc thermal conductivity for zero electric current and in general $\kappa(k) \neq \kappa_{dc}^{ij} k^i k^j$. We also define the following susceptibilities:

$$X = \frac{\partial \rho_{\mathrm{tot}}}{\partial \bar{\mu}_0}, \qquad \Xi = \frac{\partial s_{\mathrm{tot}}}{\partial \bar{\mu}_0} = \frac{\partial \rho_{\mathrm{tot}}}{\partial \bar{T}_0},$$
$$C_\rho = \oint \sqrt{h} c_{\mu 0} - \frac{\bar{T}_0 \Xi^2}{X}. \tag{3.45}$$

Note that if we consider the susceptibility $c_\rho = T(\partial s/\partial T)_\rho = c_\mu - \frac{T\xi^2}{\chi}$, in general $C_\rho \neq \oint \sqrt{h} c_{\rho 0}$. Using these definitions, we then find that

$$i\omega_+^{(2)} i\omega_-^{(2)} = \frac{\kappa(k)}{C_\rho} \frac{\sigma(k)}{X},$$
$$i\omega_+^{(2)} + i\omega_-^{(2)} = \frac{\kappa(k)}{C_\rho} + \frac{\sigma(k)}{X} + \frac{\bar{T}_0 (X\alpha(k) - \Xi\sigma(k))^2}{C_\rho X^2 \sigma(k)}. \tag{3.46}$$

This is the main result of this section and it should be compared with the general result given in (2.44), (2.45) that we obtained in the previous section.

A number of comments are in order. First, for relativistic hydrodynamics without a $U(1)$ current, there is just a single energy diffusion mode. In this case, the leading order dispersion relation is given by

$$i\omega^{(2)} = \frac{\kappa_{dc}^{ij} k_i k_j}{\bar{T}_0 \frac{\partial s_{\mathrm{tot}}}{\partial \bar{T}_0}}. \tag{3.47}$$

This result should be compared with (2.34). Similarly, we can also consider charge neutral backgrounds which have $\Xi = \alpha_{dc}^{ij} = 0$ and then the two equations in Eq. (3.41) decouple. In particular we find a charge diffusion mode with leading order dispersion relation given by

$$i\omega^{(2)} = \frac{\sigma_{dc}^{ij} k_i k_j}{\frac{\partial \rho_{\mathrm{tot}}}{\partial \bar{\mu}_0}}. \tag{3.48}$$

Our next comment concerns perturbative lattices. By definition a perturbative lattice is one in which the metric and gauge field deformations have a perturbatively small amplitude. In this case the spatial momentum dissipation is weak. Using the memory matrix formalism [33] or holography [32] we have

$$\bar{\kappa}_{dc}^{ij} = 4\pi s_0 T_0 L_{ij}^{-1}, \qquad \alpha_{dc}^{ij} = 4\pi \rho_0 L_{ij}^{-1},$$
$$\sigma_{dc}^{ij} = 4\pi s_0^{-1} \rho_0^2 L_{ij}^{-1}. \tag{3.49}$$

Here the matrix L_{ij} incorporates the leading order dissipation and $L_{ij} \to 0$ when translation invariance is retained. While all of these dc conductivities are large, κ_{dc}^{ij} and also κ in (3.44) are parametrically smaller as pointed out in [31,34]. Thus, from (3.46) we deduce that one of the frequencies will be proportional to L^{-1} while the other will be parametrically smaller.

1. Reduced hydrodynamics

When translations are broken, it should also be possible to construct a "reduced" hydrodynamical description that just involves the conserved charges i.e. the heat and the $U(1)$ charge. At the level of linear response, this can be done, in principle, by solving for $\delta u_i^{(n)}$ order by order in Eq. (3.23), to, effectively, get a set of linear equations for the variables δT and $\delta\mu$ and highly nonlocal in terms of the background metric and gauge field. We will not carry out this in any detail here, but instead highlight some interesting features of the leading order terms that would arise. In particular, we will be able to derive a set of reduced hydrodynamical equations, at the level of linear response, that generalize those discussed in [2].

We begin with the on-shell expressions for the currents in the ε expansion given in (3.34). Focusing on the $U(1)$ current for the moment, we recall that at each order $\sqrt{h} f J^{i(n)}$ are periodic functions of the x^i. We have seen that at leading order they are determined by the system of linear equations given in (3.37), which is equivalent to the system of Eqs. (3.28) that appeared for the calculation of the dc conductivity if we identify $\bar{E}_i \leftrightarrow -ik_i f\delta\mu^{(0)}$, $\bar{\zeta}_i \leftrightarrow -ik_j T_0^{-1} \delta T^{(0)}$. We can therefore write $\sqrt{h} f J^{i(1)}$ linearly in terms of $f\delta\mu^{(0)}$, $f\delta T^{(0)}$ as a sum of a constant flux, expressed in terms of the dc conductivity matrix, and a term which is co-closed and has vanishing zero mode (a periodic magnetisation current). Thus, we can write for the full current

$$\sqrt{h}fJ^i = e^{-i\omega t}e^{i\varepsilon k_i x^i}\varepsilon[(\sigma_{dc}^{ij} + \partial_k S^{kij})(-ik_j f\delta\mu^{(0)}) + (\alpha_{dc}^{ij} + \partial_k A^{kij})(-ik_j f\delta T^{(0)}) + \mathcal{O}(\varepsilon)], \tag{3.50}$$

where $S^{kij} = -S^{ikj}$, $A^{kij} = -A^{ikj}$ and both are periodic functions of the spatial coordinates. We can also obtain a similar expression for the heat current and we can write both of them in the following suggestive form:

$$\begin{aligned}\sqrt{h}fJ^i &= -(\sigma_{dc}^{ij} + \partial_k S^{kij})\nabla_j\delta\hat{\mu} - (\alpha_{dc}^{ij} + \partial_k A^{kij})\nabla_j\delta\hat{T} + \cdots,\\ \sqrt{h}fQ^i &= -\bar{T}_0(\alpha_{dc}^{ij} + \partial_k A^{kij})\nabla_j\delta\hat{\mu} - (\bar{\kappa}_{dc}^{ij} + \partial_k K^{kij})\nabla_j\delta\hat{T} + \cdots,\end{aligned} \tag{3.51}$$

where $\delta\hat{\mu} \equiv e^{-i\omega t}e^{i\varepsilon k_i x^i} f\delta\mu^{(0)}$, $\delta\hat{T} \equiv e^{-i\omega t}e^{i\varepsilon k_i x^i} f\delta T^{(0)}$ and $K^{kij} = -K^{ikj}$. In these on-shell expressions ω is fixed as an expansion in ε in terms of k_i and the background quantities via the dispersion relations.

We next consider analogous expressions for the local charge density and heat density. From (3.20) we obtain

$$\begin{aligned}\sqrt{h}fJ^t &= \sqrt{h}\rho_0 + e^{-i\omega t}e^{i\varepsilon k_i x^i}\sqrt{h}[\xi_0\delta T^{(0)} + \chi_0\delta\mu^{(0)} + \mathcal{O}(\varepsilon)],\\ \sqrt{h}fQ^t &= \sqrt{h}f(\epsilon_0 - \mu_0\rho_0) + e^{-i\omega t}e^{i\varepsilon k_i x^i}\sqrt{h}f \times [c_{\mu 0}\delta T^{(0)} + T_0\xi_0\delta\mu^{(0)} + \mathcal{O}(\varepsilon)],\end{aligned} \tag{3.52}$$

where $\sqrt{h}\rho_0$ and $\sqrt{h}f(\epsilon_0 - \mu_0\rho_0)$ are the local charge densities in equilibrium. Hence, for the perturbation we can write

$$\begin{aligned}\delta[\sqrt{h}fJ^t] &= \sqrt{h}f^{-1}\xi_0\delta\hat{T} + \sqrt{h}f^{-1}\chi_0\delta\hat{\mu} + \cdots,\\ \delta[\sqrt{h}fQ^t] &= \sqrt{h}c_{\mu 0}\delta\hat{T} + T_0\sqrt{h}\xi_0\delta\hat{\mu} + \cdots.\end{aligned} \tag{3.53}$$

At this stage, from these on-shell expressions, we now can see the leading order structure of an off-shell reduced hydrodynamics. Specifically, if we take (3.53) to be expressions for the local charge densities and (3.51) to be the associated constitutive relations for the currents, the continuity equations $\nabla_\mu J^\mu = \nabla_\mu Q^\mu = 0$ at order ε^2 will lead to the same diffusive solutions that we had above with exactly the same dispersion relations for the diffusion modes. In particular, the magnetization currents in (3.51) do not play a role in this specific calculation. It is also worth emphasizing that in this reduced hydrodynamics, the variables $\delta\hat{T}$, $\delta\hat{\mu}$ need not be periodic functions and indeed they are not in the diffusive solutions.

We can now compare these results with the hydrodynamics described in the "Methods" section of [2], highlighting several differences. First, the constitutive relations for the local currents given in [2] were declared to be given in terms of the dc conductivity, whereas here we have derived them from the underlying relativistic hydrodynamics. Second, the possibility of the terms involving S^{kij}, A^{kij}, K^{kij} was not considered in [2]. Finally, the expression for the local charge densities in [2] were not of the form (3.53). To make a connection we note that using (3.15), (3.16) we can rewrite (3.53) in the form

$$\begin{aligned}\delta[\sqrt{h}fJ^t] &= \left(\frac{\partial\rho_{\rm tot}}{\partial\bar{T}_0} + \cdots\right)\delta\hat{T} + \left(\frac{\partial\rho_{\rm tot}}{\partial\bar{\mu}_0} + \cdots\right)\delta\hat{\mu} + \cdots,\\ \delta[\sqrt{h}fQ^t] &= \left(\bar{T}_0\frac{\partial s_{\rm tot}}{\partial\bar{T}_0} + \cdots\right)\delta\hat{T} + \left(\bar{T}_0\frac{\partial s_{\rm tot}}{\partial\bar{\mu}_0} + \cdots\right)\delta\hat{\mu} + \cdots,\end{aligned} \tag{3.54}$$

where in the bracketed terms we have just written the constant zero mode part of the relevant term. The expressions (3.54) are what were considered in [2]; while the neglected higher Fourier modes will not affect the calculation of the dispersion relations for the diffusive modes, they are the same order in the ε expansion with the zero modes and they should be included as they will affect other calculations.

2. Green's functions

Within the context of relativistic hydrodynamics, the leading order solutions for the charge density and the currents are given in the previous subsection. It is possible to relate these expressions to the retarded Green's functions. At a first pass this seems problematic as the diffusive solutions are source free solutions and yet to extract Green's functions we need to relate a response to a source.

This puzzle can be resolved by the following trick. We view the solutions as having arisen after adiabatically switching on sources for the charge density in the far past, switching them off at time $t = 0$ and then comparing the solutions for $t > 0$ in the long-wavelength limit. As this is somewhat technical we have explained how this can be achieved, as well as presenting some results of general validity, in Appendix B. For simplicity, we will carry out the analysis just for the case when there is only a single current present, which is the heat current. Hence, for convenience we present the perturbed part of the diffusive solution in this case here:

$$\begin{aligned}\delta[\sqrt{h}fQ^t] &= e^{-i\omega t}e^{i\varepsilon k_i x^i}[\sqrt{h}c_{\mu 0}f\delta T^{(0)} + \mathcal{O}(\varepsilon)],\\ \sqrt{h}fQ^i &= e^{-i\omega t}e^{i\varepsilon k_i x^i}\varepsilon[(\bar{\kappa}_{dc}^{ij} + \partial_k K^{kij})(-ik_j)f\delta T^{(0)} + \mathcal{O}(\varepsilon)],\end{aligned} \tag{3.55}$$

with $i\omega = \frac{\kappa_{dc}^{ij}k_i k_j}{\bar{T}_0\frac{\partial s_{\rm tot}}{\partial\bar{T}_0}}$. We also recall that $f\delta T^{(0)}$ is constant and $\sqrt{h}c_{\mu 0}$ is a local susceptibility whose constant zero mode piece is $\bar{T}_0\frac{\partial s_{\rm tot}}{\partial\bar{T}_0}$.

IV. FINAL COMMENTS

In this paper we have made a general study of the hydrodynamical diffusion modes associated with conserved charges that arise in inhomogeneous media with a lattice symmetry. When the dc conductivities are finite, we showed that there are diffusive modes with dispersion relations that are determined by the dc conductivities and certain thermodynamical susceptibilities. This constitutes a generalized Einstein relation for inhomogeneous media. We also illustrated the general results, obtained by an analysis of retarded Green's functions, by considering the specific context of relativistic hydrodynamics. For simplicity, here we have focused on systems that are invariant under time reversal. However, it should be straightforward to generalize to the nonstatic case, after identifying suitably defined transport currents as in [20,35–38].

In [14], for a general conformal field theory on a curved manifold with a metric of the form (3.1) with $f = 1$, $h_{ij} = \Phi\delta_{ij}$ and Φ a periodic function, the relativistic hydrodynamic equations [with vanishing $U(1)$ fields] were solved for the local temperature and heat current, at the level of linear response, after applying a dc thermal gradient $\tilde{\zeta}_i$. In particular, it was shown that thermal backflow can occur whereby the heat current is locally flowing in the opposite direction to the dc source. These results can be recast in terms of the diffusion results of this paper. Let $\omega^{(2)}$ be the leading order dispersion relation as in (3.47). Then, focusing on real variables, we have leading order diffusing solutions with $\delta T = e^{-\varepsilon^2\omega^{(2)}t}\cos(\varepsilon k_i x^i)(\delta T^{(0)} + \varepsilon\delta T^{(1)} + \mathcal{O}(\varepsilon^2))$, and the local heat current given by $\delta Q^i = e^{-\varepsilon^2\omega^{(2)}t}\sin(\varepsilon k_i x^i)\varepsilon(\delta Q^{i(1)} + \mathcal{O}(\varepsilon))$, where $\delta T^{(1)}$ and $\delta Q^{i(1)}$ are the local temperature and heat current obtained in [14] for a dc thermal gradient given by $\tilde{\zeta}_i = k_i\delta T^{(0)}$. We can consider these solutions as having been adiabatically prepared in an initial state at $t = 0$ (say) and then diffusing. The solution shows that in each individual spatial period there is an elaborate local structure, which includes thermal backflow, with an overall damping of the current in time.

The existence of the same backflow current patterns that emerge in the steady state setup provides a nontrivial test of the validity of hydrodynamics for certain strongly correlated systems of electrons for which backflows have been observed. Finally, we note that the initial conditions at $t = 0$ that we are considering, arising from the construction of specific long-wavelength diffusion modes, might seem fine tuned. However, as long as short wavelength modes die out faster in time, the diffusive modes will capture the universal late-time behavior for generic initial conditions. For systems with light, spatially modulated modes, this will be case provided we examine long enough wavelengths.

The general results of this paper should also manifest themselves within the context of holography. In particular, it should be possible to obtain the Einstein relations in terms of the dc conductivities and the thermodynamic susceptibilities. It is now understood how, in general, the thermoelectric dc conductivity of the boundary field theory, when finite, can be obtained in terms of data on the black hole horizon [32,34,37]. Thus, providing one can obtain the susceptibilities in terms of horizon data, one should also be able to extract the Einstein relations. This will be explored in [39]. This line of investigation could also make contact with the recent work on relating diffusion to a characteristic velocity extracted from the black hole horizon, related to out of time ordered correlators [40,41] and [13,42–50].

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Note added.—While writing up this work, Ref. [18] appeared which also generalizes [14] to include a conserved $U(1)$ charge and independently derived the hydrodynamic Eqs. (3.23), for the special case of no time dependence and for curved manifolds with a unit norm timelike Killing vector (i.e. $f = 1$).

APPENDIX A: GENERAL RESULTS

Here we present some general results for Green's functions involving a single conserved current density operator J^μ satisfying the continuity equation $\partial_\mu J^\mu = 0$. We will present results for $G_{J^\mu J^\nu}(\omega, \mathbf{k}, \mathbf{k}')$; using the crystallographic decomposition (2.7) we can easily extract analogous results for the $G^{(\{n_j\})}_{J^\mu J^\nu}(\omega, \mathbf{k})$.

From (2.6) the current conservation condition $\partial_\mu J^\mu = 0$ implies

$$-i\omega G_{J^t B}(\omega, \mathbf{k}, \mathbf{k}') + i\mathbf{k}_i G_{J^i B}(\omega, \mathbf{k}, \mathbf{k}') = 0, \tag{A1}$$

for any operator B, whose equal time commutator with J^t vanishes. From (A1) we have

$$\begin{aligned} &-i\omega G_{J^t J^t}(\omega, \mathbf{k}, \mathbf{k}') + i\mathbf{k}_i G_{J^i J^t}(\omega, \mathbf{k}, \mathbf{k}') = 0, \\ &-i\omega G_{J^t J^j}(\omega, \mathbf{k}, \mathbf{k}') + i\mathbf{k}_i G_{J^i J^j}(\omega, \mathbf{k}, \mathbf{k}') = 0. \end{aligned} \tag{A2}$$

We next consider the time reversal invariance conditions (2.17). Since $\epsilon_{J^t} = +1$ and $\epsilon_{J^i} = -1$, we obtain

$$G_{J^iJ^t}(\omega,\mathbf{k},\mathbf{k}')=-G_{J^tJ^i}(\omega,-\mathbf{k}',-\mathbf{k}),$$
$$G_{J^iJ^j}(\omega,\mathbf{k},\mathbf{k}')=G_{J^jJ^i}(\omega,-\mathbf{k}',-\mathbf{k}). \tag{A3}$$

Combing (A3) with (A2) we therefore have

$$\mathbf{k}_i\mathbf{k}'_jG_{J^iJ^j}(\omega,\mathbf{k},\mathbf{k}')=-(i\omega)^2G_{J^tJ^t}(\omega,\mathbf{k},\mathbf{k}'),$$
$$i\mathbf{k}'_jG_{J^iJ^j}(\omega,\mathbf{k},\mathbf{k}')=(i\omega)G_{J^iJ^t}(\omega,\mathbf{k},\mathbf{k}'). \tag{A4}$$

Define the static susceptibility

$$\lim_{\omega\to0+i0}G_{J^tJ^t}(\omega,\mathbf{k},\mathbf{k}')\equiv-\chi_{J^tJ^t}(\mathbf{k},\mathbf{k}'). \tag{A5}$$

We see that (A2) and (A4) imply

$$\mathbf{k}_i\chi_{J^iJ^t}(\mathbf{k},\mathbf{k}')=0,$$
$$\mathbf{k}_i\mathbf{k}'_j\chi_{J^iJ^j}(\mathbf{k},\mathbf{k}')=0. \tag{A6}$$

Note that the sign in (A5) is fixed as follows. From (2.5), for a time-independent source for the charge density $\delta h_{J^t}(\mathbf{x})$, we have

$$\delta\langle J^t\rangle(t,\mathbf{k})=\frac{1}{(2\pi)^d}\int d\mathbf{k}'G_{J^tJ^t}(\omega=0,\mathbf{k},\mathbf{k}')\delta h_{J^t}(\mathbf{k}'). \tag{A7}$$

On the other hand from (2.4) $\delta H=(2\pi)^{-d}\int d\mathbf{k}\delta h_{J^t}(-\mathbf{k})\delta J^t(\mathbf{k})$ and so we identify the perturbed chemical potential, $\delta\mu(\mathbf{k})$, as $\delta\mu(\mathbf{k})=-\delta h_{J^t}(\mathbf{k})$. Since the static susceptibility $\chi_{J^tJ^t}$ is defined by varying the charge density with respect to the chemical potential we get the sign as in (A5).

APPENDIX B: LINEAR RESPONSE FROM A PREPARED SOURCE

We consider a perturbative deformation of the Hamiltonian as in (2.4), with a prepared source that is switched off at $t=0$, given by

$$h_B(t,\mathbf{x})=\begin{cases}e^{\varepsilon_t t+i\mathbf{k}_s\mathbf{x}}\delta h_B, & t\le0\\ 0 & t>0\end{cases}, \tag{B1}$$

with $\varepsilon_t>0$. This source contains a single spatial Fourier mode and we will be interested in taking the adiabatic limit $\varepsilon_t\to0^+$.

The time dependent expectation value of an operator A is given by the retarded Green's function as in (2.5). Thus, at $t=0$, when the sources are switched off, we have

$$\begin{aligned}\delta\langle A\rangle(t=0,\mathbf{x})&=\int dt'd\mathbf{x}'G_{AB}(-t',\mathbf{x},\mathbf{x}')\delta h_B(t',\mathbf{x}'),\\&=\int dt'd\mathbf{x}'G_{AB}(t',\mathbf{x},\mathbf{x}')e^{-\varepsilon_t t'+i\mathbf{k}_s\mathbf{x}'}\delta h_B,\\&=G_{AB}(i\varepsilon_t,\mathbf{x},\mathbf{k}_s)\delta h_B.\end{aligned} \tag{B2}$$

In the $\varepsilon_t\to0^+$ limit, after a Fourier transform, we have

$$\delta\langle A\rangle(t=0,\mathbf{k})=-\chi_{AB}(\mathbf{k},\mathbf{k}_s)\delta h_B, \tag{B3}$$

where $\chi_{AB}(\mathbf{k},\mathbf{k}')\equiv-\lim_{\omega\to0+i0}G_{AB}(\omega,\mathbf{k},\mathbf{k}')$. Also, after a Fourier transform of the source (B1), for any $t>0$ we deduce that

$$\delta\langle A\rangle(t,\mathbf{x})=\frac{1}{2\pi}\int_{-\infty}^{+\infty}d\omega\frac{1}{\varepsilon_t+i\omega}e^{-i\omega t}G_{AB}(\omega,\mathbf{x},\mathbf{k}_s)\delta h_B. \tag{B4}$$

Taking a Laplace transform in time we get

$$\begin{aligned}\delta\langle A\rangle(z,\mathbf{x})&\equiv\int_0^{+\infty}dt\delta\langle A\rangle(t,\mathbf{x})e^{izt},\\&=-\frac{1}{2\pi}\int_{-\infty}^{+\infty}d\omega\frac{1}{\omega-i\varepsilon_t}\frac{1}{\omega-z}G_{AB}(\omega,\mathbf{x},\mathbf{k}_s)\delta h_B,\end{aligned} \tag{B5}$$

with, necessarily, $\mathrm{Im}z>0$ in order for the integrals to converge. Performing a contour integral on the above expression by closing it in the upper half plane and assuming that the Green's function vanishes fast enough for large ω, we just pick up contributions from the poles at $\omega=i\varepsilon_t$ and $\omega=z$ to obtain

$$\begin{aligned}\delta\langle A\rangle(z,\mathbf{x})=&-\frac{i}{i\varepsilon_t-z}G_{AB}(i\varepsilon_t,\mathbf{x},\mathbf{k}_s)\delta h_B\\&-\frac{i}{z-i\varepsilon_t}G_{AB}(z,\mathbf{x},\mathbf{k}_s)\delta h_B.\end{aligned} \tag{B6}$$

Thus, in the $\varepsilon_t\to0^+$ limit we conclude that the spatial Fourier transform is given by (B1):

$$\delta\langle A\rangle(z,\mathbf{k})=\frac{1}{iz}(G_{AB}(z,\mathbf{k},\mathbf{k}_s)+\chi_{AB}(\mathbf{k},\mathbf{k}_s))\delta h_B. \tag{B7}$$

Using (B3) we now obtain the following solution to the initial value problem that is sourced by (B1) in the $\varepsilon_t\to0^+$ limit:

$$\begin{aligned}\delta\langle A\rangle(z,\mathbf{k})=&-\frac{1}{iz}(G_{AB}(z,\mathbf{k},\mathbf{k}_s)\chi^{-1}_{BC}(\mathbf{k},\mathbf{k}_s)\\&+\delta_{AC})\delta\langle C\rangle(t=0,\mathbf{k}).\end{aligned} \tag{B8}$$

1. Conserved current

Let us now apply some of these results to conserved currents. For simplicity we just consider the case of a single conserved current and assume that there is a single diffusion pole. We will assume that the source (B1) is a source just for the charge density operator J^t. In particular at $t = 0$ we write the source as $e^{i\mathbf{k}_s\mathbf{x}}\delta h^{(0)}_{J^t}$, with constant $\delta h^{(0)}_{J^t}$. We take the limit $\varepsilon_t \to 0$ and then consider $\mathbf{k}_s \to 0$.

From (B6), the time dependence of the charge density for $t > 0$ is fixed by the Laplace transformed quantity

$$\delta\langle J^t\rangle(z,\mathbf{x}) = e^{i\mathbf{k}_s\mathbf{x}}\sum_{\{n_j\}} e^{in_j\mathbf{k}_L^j\mathbf{x}}\frac{1}{iz}\left[G^{(\{n_j\})}_{J^tJ^t}(z,\mathbf{k}_s)+\chi^{(\{n_j\})}_{J^tJ^t}(\mathbf{k}_s)\right] \times \frac{1}{(2\pi)^d}\delta h^{(0)}_{J^t}, \tag{B9}$$

where $\chi^{(\{n_j\})}_{J^tJ^t}(\mathbf{k}_s) = -\lim_{\omega\to 0+i0} G^{(\{n_j\})}_{J^tJ^t}(\omega,\mathbf{k}_s)$. It is interesting to now examine the zero mode of the periodic function inside the sum [see (2.19)]:

$$\delta\langle J^t\rangle^{\{(0)\}}(z) = \frac{1}{iz}[G_{J^tJ^t}(z,\mathbf{k}_s) + \chi_{J^tJ^t}(\mathbf{k}_s)]\frac{1}{(2\pi)^d}\delta h^{(0)}_{J^t}, \tag{B10}$$

since we can draw some further general conclusions using the results of Sec. II A. Indeed after considering $\mathbf{k}_s \to 0$, and recalling the general results (2.30) and (2.33), we have

$$\delta\langle J^t\rangle^{\{(0)\}}(z) = \frac{-1}{-iz + \mathbf{k}_{si}\mathbf{k}_{sj}\sigma^{ij}(z)\chi(\mathbf{0})^{-1}}\chi(\mathbf{0})\frac{1}{(2\pi)^d}\delta h^{(0)}_{J^t}. \tag{B11}$$

Taking the inverse Laplace transform and keeping just the time dependence that is leading order in $\mathbf{k}_s$, we obtain

$$\delta\langle J^t\rangle^{\{(0)\}}(t) = -e^{-i\omega(\mathbf{k}_s)t}\chi(\mathbf{0})\frac{1}{(2\pi)^d}\delta h^{(0)}_{J^t}, \tag{B12}$$

with $i\omega(\mathbf{k}_s) = \sigma^{ij}_{dc}\mathbf{k}_{si}\mathbf{k}_{sj}\chi(\mathbf{0})^{-1}$.

We can now make a comparison with the diffusive solutions given in (3.55) that we found within the context of relativistic hydrodynamics. Recalling that in this appendix, and also in Sec. II, we are considering current densities, whereas in Sec. III we used current vectors, we therefore should compare the local current $\delta[\sqrt{h}fQ^t(t,\mathbf{x})]$ in (3.55) with $\delta\langle J^t\rangle(t,\mathbf{x})$. Identifying the constant source $\frac{1}{(2\pi)^d}\delta h^{(0)}_{J^t}$ here with $-f\delta T^{(0)}$ [see the discussion following (A7)], after comparing (3.55) with (B9) and the above analysis, we conclude that for these particular solutions we have that for each $\{n_j\}$, in the limit that $\mathbf{k}_s \to 0$,

$$G^{(\{n_j\})}_{J^tJ^t}(\omega,\mathbf{k}_s)\chi^{(\{n_j\})}_{J^tJ^t}(\mathbf{k}_s)^{-1} + 1 \to \frac{1}{-i\omega + \mathbf{k}_{si}\mathbf{k}_{sj}S^{\{n_j\}ij}(\omega)\chi(\mathbf{0})^{-1}}, \tag{B13}$$

with $S^{\{n_j\}ij(\omega)} = \sigma^{ij}_{dc} + \mathcal{O}(\omega)$, in order to get the correct time dependence. In particular, all of these modes of the Green's function have the same diffusion pole at the origin.

We next consider the spatial components of the current. Starting with (B8) and using (A4) we can write

$$\delta\langle J^i\rangle(z,\mathbf{k}) = \left[\frac{1}{(iz)^2}G_{J^iJ^j}(z,\mathbf{k},\mathbf{k}_s)(-i\mathbf{k}_{sj}) + \frac{1}{iz}\chi_{J^iJ^t}(\mathbf{k},\mathbf{k}_s)\right]\delta h^{(0)}_{J^t}. \tag{B14}$$

After a Fourier transform on the spatial coordinates we can therefore write

$$\delta\langle J^i\rangle(z,\mathbf{x}) = e^{i\mathbf{k}_s\mathbf{x}}\sum_{\{n_j\}} e^{in_j\mathbf{k}_L^j\mathbf{x}}\left[\frac{1}{(iz)^2}G^{(\{n_j\})}_{J^iJ^j}(z,\mathbf{k}_s)(-i\mathbf{k}_{sj}) + \frac{1}{iz}\chi^{(\{n_j\})}_{J^iJ^t}(\mathbf{k}_s)\right]\frac{1}{(2\pi)^d}\delta h^{(0)}_{J^t}. \tag{B15}$$

Current conservation implies that $\mathbf{k}_i\chi_{J^iJ^t}(\mathbf{k},\mathbf{k}_s) = 0$ [see (A6)] but in general $\chi_{J^iJ^t}(\mathbf{k},\mathbf{k}_s) \neq 0$. However, in the relativistic hydrodynamics in the static background we do have $\chi_{J^iJ^t}(\mathbf{k},\mathbf{k}_s) = 0$ [see the comment below (3.12)]. Thus, comparing (B15) with (3.55) we deduce that for the relativistic hydrodynamics, as $\mathbf{k}_s \to 0$ we have

$$\frac{1}{(i\omega)^2}G^{(\{n_j\})}_{J^iJ^j}(\omega,\mathbf{k}_s)(-i\mathbf{k}_{sj}) \to \frac{(\bar{\kappa}^{ij}_{dc} + \partial_k K^{kij})^{(\{n_j\})}(-i\mathbf{k}_{sj})}{-i\omega + \mathbf{k}_{si}\mathbf{k}_{sj}\tilde{S}^{\{n_j\}ij}(\omega)\chi(\mathbf{0})^{-1}}, \tag{B16}$$

with $\tilde{S}^{\{n_j\}ij(\omega)} = \sigma^{ij}_{dc} + \mathcal{O}(\omega)$ in order to get the correct time dependence.

A final comment is that if we consider (B3) with $\chi_{J^iJ^t}(\mathbf{k},\mathbf{k}_s) = 0$ then we deduce that $\delta\langle J^i\rangle(t=0,\mathbf{x}) = 0$. This seems inconsistent with the $t = 0$ limit of the diffusive solution arising from hydrodynamics. The resolution of this puzzle is that when we take the limit $\varepsilon_t \to 0$ it leads to a discontinuity in the current. The correct thing to do is compare the currents for $t > 0$ as we did above.

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